Incerteza

\documentclass[10pt]{article}

\usepackage[utf8]{inputenc}

\usepackage[T1]{fontenc}

\usepackage{amsmath}

\usepackage{amsfonts}

\usepackage{amssymb}

\usepackage[version=4]{mhchem}

\usepackage{stmaryrd}

\usepackage{bbold}

\title{Expected Utility Theory }

\author{}

\date{}

\begin{document}

\maketitle

These are Alexander Wolitzky's MIT notes (14.121), slightly altered by Pedro Hemsley (IE-UFRJ)

Course so far introduced basic theory of choice and utility, extended to DM and producer theory.

Last topic extends in another direction: choice under uncertainty

\section\*{Choice under Uncertainty}

All choices made under some kind of uncertainty.

Sometimes useful to ignore uncertainty, focus on ultimate choices.

Other times, must model uncertainty explicitly.

Examples:

\begin{itemize}

\item Insurance markets.

\item Financial markets.

\item Game theory.

\end{itemize}

\section\*{Overview}

Impose extra assumptions on basic choice model of Lectures 1-2.

\section\*{Rather than choosing outcome directly, decision-maker chooses uncertain prospect (or lottery).}

A lottery is a probability distribution over outcomes.

Leads to von Neumann-Morgenstern expected utility model.

\section\*{Consequences and Lotteries}

Two basic elements of expected utility theory: consequences (or outcomes) and lotteries.

\section\*{Consequences}

Finite set $C$ of consequences.

Consequences are what the decision-maker ultimately cares about.

Example: "I have a car accident, my insurance company covers most of the costs, but I have to pay a $\$ 500$ deductible."

\section\*{Decision-maker (DM) does not choose consequences directly.}

Lotteries

DM chooses a lottery, $p$.

Lotteries are probability distributions over consequences:

$p: C \rightarrow[0,1]$ with $\sum\_{c \in C} p(c)=1$.

Set of all lotteries is denoted by $P$.

Example: "A gold-level health insurance plan, which covers all kinds of diseases, but has a $\$ 500$ deductible."

Makes sense because DM assumed to rank health insurance plans only insofar as lead to different probability distributions over consequences.

\section\*{Choice}

Decision-maker makes choices from set of alternatives $X$.

What's set of alternatives here, $C$ or $P$ ?

Answer: $P$

\section\*{DM does not choose consequences directly, but instead chooses lotteries.}

Assume decision-maker has a rational preference relation $\gtrsim$ on $P$.

Natural to assume this?

\section\*{Convex Combinations of Lotteries}

Given two lotteries $p$ and $p^{\prime}$, the convex combination $\alpha p+(1-\alpha) p^{\prime}$ is the lottery defined by

$$

\left(\alpha p+(1-\alpha) p^{\prime}\right)(c)=\alpha p(c)+(1-\alpha) p^{\prime}(c) \text { for all } c \in C

$$

One way to generate it:

\begin{itemize}

\item First, randomize between $p$ and $p^{\prime}$ with weights $\alpha$ and $1-\alpha$.

\item Second, choose a consequence according to whichever lottery came up.

\end{itemize}

Such a probability distribution over lotteries is called a compound lottery.

In expected utility theory, no distinction between simple and compound lotteries: simple lottery $\alpha p+(1-\alpha) p^{\prime}$ and above compound lottery give same distribution over consequences, so identified with same element of $\boldsymbol{P}$.

So, no problem if DM doesn't know exactly the distribution for something. We'll come back to this.

\section\*{The Set $P$}

As $\alpha p+(1-\alpha) p^{\prime}$ is also a lottery, $P$ is convex.

$P$ is also closed and bounded (why?).

$\Rightarrow P$ is a compact subset of $\mathbb{R}^{n}$, where $n=|C|$.

Whenever $\succsim$ is rational and continuous, can be represented by continuous utility function $U: P \rightarrow \mathbb{R}$ :

$$

p \gtrsim q \Leftrightarrow U(p) \geq U(q)

$$

We're just applying it to lotteries because that's what the DM chooses now.

Intuitively, want more than this.

Want not only that DM has utility function over lotteries, but also that somehow related to "utility" over consequences.

Only care about lotteries insofar as affect distribution over consequences, so preferences over lotteries should have something to do with "preferences" over consequences.

\section\*{Expected Utility}

Best we could hope for is representation by utility function of following form:

Definition: a utility function $U: P \rightarrow \mathbb{R}$ has an expected utility form if there exists a function $u: C \rightarrow \mathbb{R}$ such that

$$

U(p)=\sum\_{c \in C} p(c) u(c) \text { for all } p \in P

$$

In this case, the function $U$ is called an expected utility function, and the function $u$ is call a von Neumann-Morgenstern utility function.

If preferences over lotteries happen to have an expected utility representation, it's as if DM has a "utility function" over consequences (and chooses among lotteries so as to maximize expected "utility over consequences").

Remarks

$$

U(p)=\sum\_{c \in C} p(c) u(c)

$$

Expected utility function $U: P \rightarrow \mathbb{R}$ represents preferences $\gtrsim$ on $P$ just as we had before

$U: P \rightarrow \mathbb{R}$ is an example of a standard utility function.

von Neumann-Morgenstern utility function $u: C \rightarrow \mathbb{R}$ is not a standard utility function.

\section\*{Can't have a "real" utility function on consequences, as DM never chooses among consequences.}

If preferences over lotteries happen to have an expected utility representation, it's as if DM has a "utility function" over consequences.

This "utility function" over consequences is the von Neumann-Morgenstern utility function.

\section\*{Example}

Suppose hipster restaurant doesn't let you order steak or chicken, but only probability distributions over steak and chicken.

How should you assess menu item ( $p$ (steak), $p$ (chicken))?

One way: ask yourself how much you'd like to eat steak, $u$ (steak), and chicken, $u$ (chicken), and evaluate according to

$$

p(\text { steak }) \cdot u(\text { steak })+p(\text { chicken }) \cdot u(\text { chicken })

$$

If this is what you'd do, then your preferences have an expected utility representation.

Suppose instead you choose whichever menu item has $p$ (steak) closest to $\frac{1}{2}$.

Your preferences are rational, so they have a utility representation.

But they do not have an expected utility representation - we'll see this.

\section\*{Property of EU: Linearity in Probabilities}

\section\*{Theorem}

If $U: P \rightarrow \mathbb{R}$ is an expected utility function, then

$$

U\left(\alpha p+(1-\alpha) p^{\prime}\right)=\alpha U(p)+(1-\alpha) U\left(p^{\prime}\right)

$$

In fact, a utility function $U: P \rightarrow \mathbb{R}$ has an expected utility form iff this equation holds for all $p, p^{\prime}$, and $\alpha \in[0,1]$.

Proof: appendix.

\section\*{Property of EU: Invariant to Affine Transformations}

Suppose $U: P \rightarrow \mathbb{R}$ is an expected utility function representing preferences $\succsim$.

Any increasing transformation of $U$ also represents $\succsim$.

Not all increasing transformations of $U$ have expected utility form.

\section\*{Theorem}

Suppose $U: P \rightarrow \mathbb{R}$ is an expected utility function representing preferences $\succsim$. Then $V: P \rightarrow \mathbb{R}$ is also an expected utility function representing $\gtrsim$ iff there exist $a, b>0$ such that

$$

V(p)=a+b U(p) \text { for all } p \in P

$$

If this is so, we also have $V(p)=\sum\_{c \in C} p(c) v(c)$ for all $p \in P$, where

$$

v(c)=a+b u(c) \text { for all } c \in C

$$

Proof: appendix.

\section\*{What Preferences have an Expected Utility Representation?}

Preferences must be rational to have any kind of utility representation.

Preferences on a compact and convex set must be continuous to have a continuous utility representation.

Besides rationality and continuity, what's needed to ensure that preferences have an expected utility representation?

\section\*{The Independence Axiom}

Definition

A preference relation $\gtrsim$ satisfies independence if, for every

$p, q, r \in P$ and $\alpha \in(0,1)$,

$$

p \succsim q \Leftrightarrow \alpha p+(1-\alpha) r \succsim \alpha q+(1-\alpha) r

$$

Can interpret as form of "dynamic consistency."

Doesn't need to hold for consequences.

\section\*{Back to Example}

Suppose choose lottery with $p$ (steak) closest to $\frac{1}{2}$.

Let $p=\left(\frac{1}{2}, \frac{1}{2}\right), q=(0,1), r=(1,0)$, and $\alpha=\frac{1}{2}$.

Then

$$

p=\left(\frac{1}{2}, \frac{1}{2}\right)>(0,1)=q

$$

but

$$

\alpha q+(1-\alpha) r=\left(\frac{1}{2}, \frac{1}{2}\right)>\left(\frac{3}{4}, \frac{1}{4}\right)=\alpha p+(1-\alpha) r

$$

Does not satisfy independence.

\section\*{Expected Utility: Characterization}

A preference relation $\gtrsim$ has an expected utility representation iff it satisfies rationality, continuity, and independence.

Intuition: both having expected utility form and satisfying independence boil down to having straight, parallel indifference curves.

\section\*{Subjective Expected Utility Theory}

So far, probabilities are objective.

In reality, uncertainty is usually subjective.

Subjective expected utility theory (Savage, 1954): under assumptions roughly similar to ones form this lecture, preferences have an expected utility representation where both the utilities over consequences and the subjective probabilities themselves are revealed by decision-maker's choices.

Thus, expected utility theory applies even when the probabilities are not objectively given.

(To learn more, a good starting point is Kreps (1988), "Notes on the Theory of Choice." )

Again, no problem if DM doesn't know the exact distribution.

The same holds in general equilibrium: allows for different individual priors.

One may go beyond and assume DM has some rule to deal with set of priors - e.g., DM may assume that nature will choose the worst possible prior, conditional on his optimal choice, leading to a mini-max structure that deals with fear of misspecification and relates to sub-rational behavior.

See nice discussion in Hansen and Sargent (2000) and a critique by Sims (AER 2001).

\section\*{Attitudes toward Risk}

\section\*{Money Lotteries}

Turn now to special case of choice under uncertainty where outcomes are measured in dollars.

Set of consequences $C$ is subset of $\mathbb{R}$.

A lottery is a cumulative distribution function $F$ on $\mathbb{R}$.

(Now we use $F$ instead of $p$ )

Assume preferences have expected utility representation:

$$

U(F)=E\_{F}[u(x)]=\int u(x) f(x) d x

$$

More generally, we could write $\int u(x) d F(x)$.

This is useful if we do not know whether a density $f$ exists.

We'll assume it does and make $d F(x) / d x=f(x)$, so that $d F(x)=f(x) d x$, leading to our representation above.

(But everything holds for a general $F(x)$.)

Assume $u$ increasing, differentiable.

Question: how do properties of von Neumann-Morgenstern utility function $u$ relate to decision-maker's attitude toward risk?

\section\*{Expected Value vs. Expected Utility}

Expected value of lottery $F$ is

$$

E\_{F}[x]=\int x f(x) d x

$$

Expected utility of lottery $F$ is

$$

E\_{F}[u(x)]=\int u(x) f(x) d x

$$

Can learn about DM's risk attitude by comparing $E\_{F}[u(x)]$ and $u\left(E\_{F}[x]\right)$.

\section\*{Risk Attitude: Definitions}

\section\*{Definition}

A decision-maker is risk-averse if she always prefers the sure wealth level $E\_{F}[x]$ to the lottery $F$ : that is,

$$

\int u(x) f(x) d x \leq u\left(\int x f(x) d x\right) \text { for all } F

$$

A decision-maker is strictly risk-averse if the inequality is strict for all nondegenerate lotteries $F$.

A decision-maker is risk-neutral if she is always indifferent:

$$

\int u(x) f(x) d x=u\left(\int x f(x) d x\right) \text { for all } F

$$

A decision-maker is risk-loving if she always prefers the lottery:

$$

\int u(x) f(x) d x \geq u\left(\int x f(x) d x\right) \text { for all } F

$$

\section\*{Risk Aversion and Concavity}

Statement that $\int u(x) d F(x) \leq u\left(\int x d F(x)\right)$ for all $F$ is called Jensen's inequality.

Fact: Jensen's inequality holds iff $u$ is concave.

This implies:

Theorem

A decision-maker is (strictly) risk-averse if and only if $u$ is (strictly) concave.

A decision-maker is risk-neutral if and only if $u$ is linear.

A decision-maker is (strictly) risk-loving if and only if $u$ is (strictly) convex.

\section\*{Certainty Equivalents}

Can also define risk-aversion using certainty equivalents.

\section\*{Definition}

The certainty equivalent of a lottery $F$ is the sure wealth level that yields the same expected utility as $F$ : that is,

$$

u[C E(F, u)]=\int u(x) f(x) d x

$$

That is,

$$

C E(F, u)=u^{-1}\left(\int u(x) d F(x)\right)

$$

Theorem

A decision-maker is risk-averse iff $C E(F, u) \leq E\_{F}(x)$ for all $F$.

A decision-maker is risk-neutral iff $C E(F, u)=E\_{F}(x)$ for all $F$.

A decision-maker is risk-loving iff $C E(F, u) \geq E\_{F}(x)$ for all $F$.

\section\*{Quantifying Risk Attitude}

We know what it means for a DM to be risk-averse.

What does it mean for one DM to be more risk-averse than another?

Two possibilities:

\begin{enumerate}

\item $u$ is more risk-averse than $v$ if, for every $F, C E(F, u) \leq C E(F, v)$.

\item $u$ is more risk-averse than $v$ if $u$ is "more concave" than $v$, in that $u=g \circ v$ for some increasing, concave $g$.

\end{enumerate}

One more, based on local curvature of utility function: $u$ is more-risk averse than $v$ if, for every $x$,

$$

-\frac{u^{\prime \prime}(x)}{u^{\prime}(x)} \geq-\frac{v^{\prime \prime}(x)}{v^{\prime}(x)}

$$

$A(x, u)=-\frac{u^{\prime \prime}(x)}{u^{\prime}(x)}$ is called the Arrow-Pratt coefficient of absolute risk-aversion.

\section\*{An Equivalence}

Theorem

The following are equivalent:

\begin{enumerate}

\item For every $F, C E(F, u) \leq C E(F, v)$.

\item There exists an increasing, concave function $g$ such that $u=g \circ v$.

\item For every $x, A(x, u) \geq A(x, v)$.

\end{enumerate}

\section\*{Risk Attitude and Wealth Levels}

How does risk attitude vary with wealth?

Natural to assume that a richer individual is more willing to bear risk: whenever a poorer individual is willing to accept a risky gamble, so is a richer individual.

Captured by decreasing absolute risk-aversion:

\section\*{Definition}

A von Neumann-Morenstern utility function $u$ exhibits decreasing (constant, increasing) absolute risk-aversion if $A(x, u)$ is decreasing (constant, increasing) in $x$.

\section\*{Risk Attitude and Wealth Levels}

Theorem

Suppose $u$ exhibits decreasing absolute risk-aversion.

If the decision-maker accepts some gamble at a lower wealth level, she also accepts it at any higher wealth level:

that is, for any lottery $F(x)$, if

$$

E\_{F}[u(w+x)] \geq u(w)

$$

then, for any $w^{\prime}>w$,

$$

E\_{F}\left[u\left(w^{\prime}+x\right)\right] \geq u\left(w^{\prime}\right)

$$

\section\*{Multiplicative Gambles}

What about gambles that multiply wealth, like choosing how risky a stock portfolio to hold? Are richer individuals also more willing to bear multiplicative risk? Depends on increasing/decreasing relative risk-aversion:

$$

R(x, u)=-\frac{u^{\prime \prime}(x)}{u^{\prime}(x)} x

$$

Theorem

Suppose u exhibits decreasing relative risk-aversion.

If the decision-maker accepts some multiplicative gamble at a lower wealth level, she also accepts it at any higher wealth level: that is, for any lottery $F(t)$, if

$$

E\_{F}[u(t w)] \geq u(w)

$$

then, for any $w^{\prime}>w$,

Relative Risk-Aversion vs. Absolute Risk-Aversion

$$

R(x)=x A(x)

$$

decreasing relative risk-aversion $\Rightarrow$ decreasing absolute risk-aversion

increasing absolute risk-aversion $\Rightarrow$ increasing relative risk-aversion

Ex. decreasing relative risk-aversion $\Rightarrow$ more willing to gamble $1 \%$ of wealth as get richer.

So certainly more willing to gamble a fixed amount of money.

\section\*{Application: Insurance}

Risk-averse agent with wealth $w$, faces probability $p$ of incurring monetary $\operatorname{loss} L$.

Can insure against the loss by buying a policy that pays out $a$ if the loss occurs.

Policy that pays out a costs qa.

How much insurance should she buy?

\section\*{Agent's Problem}

$$

\max \_{a} p u(w-q a-L+a)+(1-p) u(w-q a)

$$

$u$ concave $\Rightarrow$ concave problem, so FOC is necessary and sufficient.

FOC:

$$

p(1-q) u^{\prime}(w-q a-L+a)=(1-p) q u^{\prime}(w-q a)

$$

Equate marginal benefit of extra dollar in each state.

\section\*{Actuarily Fair Prices}

Insurance is actuarily fair if expected payout qa equals cost of insurance $p a$ : that is, $p=q$.

With acturarily fair insurance, FOC becomes

$$

u^{\prime}(w-q a-L+a)=u^{\prime}(w-q a)

$$

Solution: $a=L$

A risk-averse DM facing actuarily fair prices will always fully insure.

\section\*{Actuarily Unfair Prices}

What if insurance company makes a profit, so $q>p$ ?

Rearrange FOC as

$$

\frac{u^{\prime}(w-q a-L+a)}{u^{\prime}(w-q a)}=\frac{(1-p) q}{p(1-q)}>1

$$

Solution: $a<L$

A risk-averse DM facing actuarily unfair prices will never fully insure.

Intuition: $u$ approximately linear for small risks, so not worth giving up expected value to insure away last little bit of variance.

\section\*{Comparative Statics}

$$

\max \_{a} p u(w-q a-L+a)+(1-p) u(w-q a)

$$

Bigger loss $\Rightarrow$ buy more insurance ( $a^{\*}$ increasing in $L$ ) Follows from Topkis' theorem.

If agent has decreasing absolute risk-aversion, then she buys less insurance as she gets richer.

Prove it as an exercise!

\section\*{Application: Portfolio Choice}

Risk-averse agent with wealth $w$ has to invest in a safe asset and a risky asset.

Safe asset pays certain return $r$.

Risky asset pays random return $z$, with $\operatorname{cdf} F$.

Agent's problem

$$

\max \_{a \in[0, w]} \int u(a z+(w-a) r) d F(z)

$$

First-order condition

$$

\int(z-r) u^{\prime}(a z+(w-a) r) d F(z)=0

$$

\section\*{Risk-Neutral Benchmark}

Suppose $u^{\prime}(x)=\alpha x$ for some $\alpha>0$.

Then

$$

U(a)=\int \alpha(a z+(w-a) r) d F(z)

$$

so

$$

U^{\prime}(a)=\alpha(E[z]-r)

$$

Solution: set $a=w$ if $E[z]>r$, set $a=0$ if $E[z]<r$.

Risk-neutral investor puts all wealth in the asset with the highest rate of return.

$r>E[z]$ Benchmark

$$

U^{\prime}(0)=\int(z-r) u^{\prime}(w) d F=(E[z]-r) u^{\prime}(w)

$$

If safe asset has higher rate of return, then even risk-averse investor puts all wealth in the safe asset.

\section\*{More Interesting Case}

What if agent is risk-averse, but risky asset has higher expected return?

$$

U^{\prime}(0)=(E[z]-r) u^{\prime}(w)>0

$$

If risky asset has higher rate of return, then risk-averse investor always puts some wealth in the risky asset.

\section\*{Comparative Statics}

Does a less risk-averse agent always invest more in the risky asset?

Sufficient condition for agent $v$ to invest more than agent $u$ :

$$

\begin{gathered}

\int(z-r) u^{\prime}(a z+(w-a) r) d F=0 \\

\Rightarrow \int(z-r) v^{\prime}(a z+(w-a) r) d F \geq 0

\end{gathered}

$$

$u$ more risk-averse $\Rightarrow v=h \circ u$ for some increasing, convex $h$. Inequality equals

$$

\int(z-r) h^{\prime}(u(a z+(w-a) r)) u^{\prime}(a z+(w-a) r) d F \geq 0

$$

$h^{\prime}(\cdot)$ positive and increasing in $z$

$\Rightarrow$ multiplying by $h^{\prime}(\cdot)$ puts more weight on positive $(z>r)$ terms, less weight on negative terms.

A less risk-averse agent always invests more in the risky=asset.

\section\*{Comparing Risky Prospects}

\section\*{Risky Prospects}

We've studied decision-maker's subjective attitude toward risk.

Now: study objective properties of risky prospects (lotteries, gambles) themselves, relate to individual decision-making.

Topics:

\begin{itemize}

\item First-Order Stochastic Dominance

\item Second-Order Stochastic Dominance

\item (Optional) Some recent research extending these concepts

\end{itemize}

\section\*{First-Order Stochastic Dominance}

When is one lottery unambiguously better than another?

Natural definition: $F$ dominates $G$ if, for every amount of money $x, F$ is more likely to yield at least $x$ dollars than $G$ is.

\section\*{Definition}

For any lotteries $F$ and $G$ over $\mathbb{R}, F$ first-order stochastically dominates (FOSD) $G$ if

$$

F(x) \leq G(x) \text { for all } x

$$

\section\*{FOSD and Choice}

Main theorem relating FOSD to decision-making:

\section\*{Theorem}

$F$ FOSD $G$ iff every decision-maker with a non-decreasing utility function prefers $F$ to $G$.

That is, the following are equivalent:

\begin{enumerate}

\item $F(x) \leq G(x)$ for all $x$.

\item $\int u(x) d F \geq \int u(x) d G$ for every non-decreasing function $u: \mathbb{R} \rightarrow \mathbb{R}$.

\end{enumerate}

Proof:

Preferred by Everyone $=>$ FOSD

If $F$ does not FOSD $G$, then there's some amount of money $x^{\*}$ such that $G$ is more likely to give at least $x^{\*}$ than $F$ is.

Consider a DM who only cares about getting at least $x^{\*}$ dollars.

She will prefer $G$.

$$

\text { FOSD }=>\text { Preferred by Everyone }

$$

Main idea: $F$ FOSD $G \Rightarrow F$ gives more money "realization-by-realization."

Suppose draw $x$ according to $G$, but then instead give decision-maker

$$

y(x)=F^{-1}(G(x))

$$

Then:

\begin{enumerate}

\item $y(x) \geq x$ for all $x$, and

\item $y$ is distributed according to $F$.

\end{enumerate}

$\Rightarrow$ paying decision-maker according to $F$ just like first paying according to $G$, then sometimes giving more money.

Any decision-maker who likes money likes this.

QED.

Second-Order Stochastic Dominance

Q : When is one lottery better than another for any decision-maker?

A: First-Order Stochastic Dominance.

Q: When is one lottery better than another for any risk-averse decision-maker?

A: Second-Order Stochastic Dominance.

\section\*{Definition}

$F$ second-order stochastically dominates (SOSD) $G$ iff every decision-maker with a non-decreasing and concave utility function prefers $F$ to $G$ : that is,

$$

\int u(x) d F \geq \int u(x) d G

$$

for every non-decreasing and concave function $u: \mathbb{R} \rightarrow \mathbb{R}$.

SOSD is a weaker property than FOSD.

SOSD for Distributions with Same Mean

If $F$ and $G$ have same mean, when will any risk-averse decision-maker prefer $F$ ?

When is $F$ "unambiguously less risky" than $G$ ?

Mean-Preserving Spreads

$G$ is a mean-preserving spread of $F$ if $G$ can be obtained by first drawing a realization from $F$ and then adding noise.

Definition

$G$ is a mean-preserving spread of $F$ iff there exist random variables $x, y$, and $\varepsilon$ such that

$$

y=x+\varepsilon

$$

$x$ is distributed according to $F, y$ is distributed according to $G$, and $E[\varepsilon \mid x]=0$ for all $x$.

Formulation in terms of cdfs:

$$

\int\_{-\infty}^{x} G(y) d y \geq \int\_{-\infty}^{x} F(y) d y \text { for all } x

$$

Characterization of SOSD for CDFs with Same Mean

\section\*{Theorem}

Assume that $\int x d F=\int x d G$. Then the following are equivalent:

\begin{enumerate}

\item F SOSD $G$.

\item $G$ is a mean-preserving spread of $F$.

\item $\int\_{-\infty}^{x} G(y) d y \geq \int\_{-\infty}^{x} F(y) d y$ for all $x$.

\end{enumerate}

General Characterization of SOSD

Theorem

The following are equivalent:

\begin{enumerate}

\item F SOSD G.

\item $\int\_{-\infty}^{x} G(y) d y \geq \int\_{-\infty}^{x} F(y) d y$ for all $x$.

\item There exist random variables $x, y, z$, and $\varepsilon$ such that

\end{enumerate}

$$

y=x+z+\varepsilon

$$

$x$ is distributed according to $F, y$ is distributed according to $G, z$ is always nonpositive, and $E[\varepsilon \mid x]=0$ for all $x$.

\begin{enumerate}

\setcounter{enumi}{3}

\item There exists a cdf $H$ such that $F$ FOSD $H$ and $G$ is a mean-preserving spread of $H$.

\end{enumerate}

\section\*{Complete Dominance Orderings [Optional]}

FOSD and SOSD are partial orders on lotteries:

"most distributions" are not ranked by FOSD or SOSD.

To some extent, nothing to be done:

If $F$ doesn't FOSD $G$, some decision-maker prefers $G$.

If $F$ doesn't SOSD $G$, some risk-averse decision-maker prefers $G$.

However, recent series of papers points out that if view $F$ and $G$ as lotteries over monetary gains and losses rather than final wealth levels, and only require that no decision-maker prefers $G$ to $F$ for all wealth levels, do get a complete order on lotteries (and index of lottery's "riskiness").

\section\*{Acceptance Dominance}

Consider decision-maker with wealth $w$, has to accept or reject a gamble $F$ over gains $/ \operatorname{losses} x$.

Accept iff

$$

E\_{F}[u(w+x)] \geq u(w)

$$

\section\*{Definition}

$F$ acceptance dominates $G$ if, whenever $F$ is rejected by decision-maker with concave utility function $u$ and wealth $w$, so is G.

That is, for all $u$ concave and $w>0$,

$$

\begin{aligned}

& E\_{F}[u(w+x)] \leq u(w) \\

& E\_{G}[u(w+x)] \leq u(w)

\end{aligned}

$$

\section\*{Acceptance Dominance and FOSD/SOSD}

$F \operatorname{SOSD} G$

$\Rightarrow E\_{F}[u(w+x)] \geq E\_{G}[u(w+x)]$ for all concave $u$ and wealth $w$

$\Rightarrow F$ acceptance dominates $G$.

If $E\_{F}[x]>0$ but $x$ can take on both positive and negative values, can show that $F$ acceptance dominates lottery that doubles all gains and losses.

Acceptance dominance refines SOSD.

But still very incomplete.

Turns out can get complete order from something like: acceptance dominance at all wealth levels, or for all concave utility functions.

\section\*{Wealth Uniform Dominance}

\section\*{Definition}

$F$ wealth-uniformly dominates $G$ if, whenever $F$ is rejected by decision-maker with concave utility function $u$ at every wealth level $w$, so is $G$.

That is, for all $u \in U^{\*}$,

$$

\begin{aligned}

& E\_{F}[u(w+x)] \leq u(w) \text { for all } w>0 \\

& E\_{G}[u(w+x)] \leq u(w) \text { for all } w>0

\end{aligned}

$$

\section\*{Utility Uniform Dominance}

\section\*{Definition}

$F$ utility-uniformly dominates $G$ if, whenever $F$ is rejected at wealth level $w$ by a decision-maker with any utility function $u \in U^{\*}$, so is $G$.

That is, for all $w>0$,

$$

\begin{aligned}

& E\_{F}[u(w+x)] \leq u(w) \text { for all } u \in U^{\*} \\

& E\_{G}[u(w+x)] \leq u(w) \text { for all } u \in U^{\*}

\end{aligned}

$$

\section\*{Uniform Dominance: Results}

Hart (2011):

\begin{itemize}

\item Wealth-uniform dominance and utility-uniform dominance are complete orders.

\item Comparison of two lotteries in these orders boils down to comparison of simple measures of the "riskiness" of the lotteries.

\item Measure for wealth-uniform dominance: critical level of risk-aversion above which decision maker with constant absolute risk-aversion rejects the lottery.

\item Measure for utility-uniform dominance: critical level of wealth below which decision-maker with log utility rejects the lottery.

\end{itemize}

\section\*{Appendix: some proofs}

\section\*{$U$ has expected utility form $\Leftrightarrow U$ linear in probabilities}

\section\*{Theorem}

$U: P \rightarrow \mathbb{R}$ has an expected utility form if and only if

$$

U\left(\alpha p+(1-\alpha) p^{\prime}\right)=\alpha U(p)+(1-\alpha) U\left(p^{\prime}\right)

$$

holds for all $p, p^{\prime}$, and $\alpha \in[0,1]$.

Notice: this is MWG proposition 6B1. It uses the following notation: $U\left(\sum \alpha\_{k} p\_{k}\right)=$ $\sum \alpha\_{k} U\left(p\_{k}\right)$, just substituting $p$ for $L$ (which stands for 'lottery').

Proof

Without loss of generality, we will assume only two consequences, $c^{1}$ and $c^{2}$.

Hence any lottery $p$ may be written as $p=\left(p^{1}, p^{2}\right)$, in which $p^{1}=\operatorname{Prob}\left(c^{1}\right)$ and $p^{2}=\operatorname{Prob}\left(c^{2}\right)$.

All arguments below hold unchanged for $p=\left(p^{1}, \ldots, p^{n}\right)$, that is, for $n$ consequences $c^{1}, \ldots, c^{n}$. This extension is shown in red below; you may simply ignore it in your first reading.

The arguments below also hold for $c \in\left[c^{1}, c^{n}\right] \in \mathbb{R}$, but the math is not exactly the same.

\section\*{Necessity: $U$ linear in probabilities $\Rightarrow U$ has expected utility form}

Write lottery $p=\left(p^{1}, p^{2}\right)$ as a convex combination of degenerate lotteries $\left(C^{1}, C^{2}\right)$ :

$$

p=p^{1} C^{1}+p^{2} C^{2}+\cdots+p^{n} C^{n}

$$

That is, $C^{1}=(1,0)$, meaning that consequence $1\left(c^{1}\right)$ happens with probability 1 , and $C^{2}=(0,1)$, meaning that consequence $2\left(c^{2}\right)$ happens with probability 1 . The\\

equation above is simply $p=\left(p^{1}, p^{2}\right)=\left(p^{1}, 0\right)+\left(0, p^{2}\right)=p^{1} \cdot(1,0)+p^{2} \cdot(0,1)=$ $p^{1} C^{1}+p^{2} C^{2}$.

Then:

$$

U(p)=U\left(p^{1} C^{1}+p^{2} C^{2}+\cdots+p^{n} C^{n}\right)=p^{1} U\left(C^{1}\right)+p^{2} U\left(C^{2}\right)+\cdots+p^{n} U\left(C^{n}\right)

$$

The second equality follows from our assumption: $U$ is linear in probabilities.

But $U\left(C^{1}\right)$ is the utility from a degenerate lottery, that is, it's simply the vNM utility of consequence $c\_{1}: U\left(C^{1}\right)=u\left(c^{1}\right)$.

Remember our notation: big U is for DM's actual utility; small u is form DM's vNM utility. Big C denotes a lottery; small c denotes a consequence.

The last equation may be rewritten as:

$$

\begin{gathered}

U(p)=U\left(p^{1} C^{1}+p^{2} C^{2}+\cdots+p^{n} C^{n}\right)=p^{1} U\left(C^{1}\right)+p^{2} U\left(C^{2}\right)+\cdots+p^{n} U\left(C^{n}\right) \\

=p^{1} u\left(c^{1}\right)+p^{2} u\left(c^{2}\right)+\cdots+p^{n} u\left(c^{n}\right)

\end{gathered}

$$

In short:

$$

U(p)=p^{1} u\left(c^{1}\right)+p^{2} u\left(c^{2}\right)+\cdots+p^{n} u\left(c^{n}\right)

$$

This is exactly the expected utility property, concluding the proof.

\section\*{Sufficiency: $U$ has expected utility form $\Rightarrow U$ linear in probabilities}

Consider a compound lottery:

$$

\left(p\_{1}, p\_{2}, \ldots, p\_{k} ; \alpha\_{1}, \alpha\_{2}, \ldots, \alpha\_{k}\right)

$$

Notice that now we have $p\_{1}$ instead of $p^{1}$; and $p\_{2}$ instead of $p^{2}$. Superscripts refer to consequences; subscripts refer to specific lotteries.

That is, $p\_{1}$ and $p\_{2}$ are different lotteries, and each one is a vector assigning probabilities for each of the two possible consequences $c^{1}$ and $c^{2}$ :

$$

p\_{i}=\left(p\_{i}^{1}, p\_{i}^{2}, \ldots, p\_{i}^{n}\right)

$$

For $i=1, \ldots, k$. That is, we have $k$ lotteries, and each one is chosen with probability $\alpha\_{i}$ in our compound lottery.

We will allow $k$ to be generic. If you want, just take $k=2$ in the following computations - again, it's without loss of generality, but be careful not to confuse the number of consequences with the number of lotteries.

Consider now the following (reduced) lottery:

$$

\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}+\cdots+\alpha\_{k} p\_{k}

$$

Consider the utility of this lottery:

$$

U\left(\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}+\cdots+\alpha\_{k} p\_{k}\right)

$$

We may now use our assumption: $U$ has the expected utility form. That is, one may rewrite this utility as:

$$

U\left(\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}+\cdots+\alpha\_{k} p\_{k}\right)=u^{1} \cdot \operatorname{Prob}\left(u^{1}\right)+u^{2} \cdot \operatorname{Prob}\left(u^{2}\right)+\cdots+u^{n} \cdot \operatorname{Prob}\left(u^{n}\right)

$$

What are these $u^{i}$ s? We just need to know that there are some $u^{i}$ s that make this equation hold - our assumption guarantees this is the case. But we do have an interpretation for them: $u^{i}$ is just the vNM utility of consequence $c^{i}$. Analogously, $\operatorname{Prob}\left(u^{i}\right)$ is simply the probability of this consequence, computed from the compound lottery $\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}+\cdots+\alpha\_{k} p\_{k}$.

This explains why we have subscripts on the LHS, but superscripts on the RHS. In the LHS, we have lotteries (that generate a compound lottery). On the RHS, we have consequences with vNM utilities $u^{1}, u^{2}, \ldots, u^{n}$. If you want, you may think of the lottery on the LHS as any given lottery $p$.

Let's develop this equation:

$$

\begin{aligned}

& U\left(\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}+\cdots+\alpha\_{k} p\_{k}\right)= \\

& u^{1} \cdot \operatorname{Prob}\left(u^{1}\right)+u^{2} \cdot \operatorname{Prob}\left(u^{2}\right)= \\

& u^{1} \cdot \underbrace{\left(\alpha\_{1} p\_{1}^{1}+\alpha\_{2} p\_{2}^{1}+\cdots+\alpha\_{k} p\_{k}^{1}\right)}\_{\operatorname{Prob}\left(u^{1}\right)}+u^{2} \cdot \underbrace{\left(\alpha\_{1} p\_{1}^{2}+\alpha\_{2} p\_{2}^{2}+\cdots+\alpha\_{k} p\_{k}^{2}\right)}\_{\operatorname{Prob}\left(u^{2}\right)}+\cdots+u^{n} \\

& \underbrace{\left(\alpha\_{1} p\_{1}^{n}+\alpha\_{2} p\_{2}^{n}+\cdots+\alpha\_{k} p\_{k}^{n}\right)}\_{\operatorname{Prob}\left(u^{2}\right)}= \\

& \alpha\_{1} \cdot \underbrace{\left(u^{1} \cdot p\_{1}^{1}+u^{2} \cdot p\_{1}^{2}+\cdots+u^{n} \cdot p\_{1}^{n}\right)}\_{U\left(p\_{1}\right)}+\alpha\_{2} \cdot \underbrace{\left(u^{1} \cdot p\_{2}^{1}+u^{2} \cdot p\_{2}^{2}+\cdots+u^{n} \cdot p\_{2}^{n}\right)}\_{U\left(p\_{2}\right)}+\cdots \\

& +\alpha\_{k} \cdot \underbrace{\left(u^{1} \cdot p\_{k}^{1}+u^{2} \cdot p\_{k}^{2}+\cdots+u^{n} \cdot p\_{k}^{n}\right)}\_{U\left(p\_{k}\right)}= \\

& \alpha\_{1} \cdot U\left(p\_{1}\right)+\alpha\_{2} \cdot U\left(p\_{2}\right)+\cdots+\alpha\_{k} \cdot U\left(p\_{k}\right)

\end{aligned}

$$

From the second to the third line, we use the definition of $\operatorname{Prob}\left(u^{1}\right)$ : it is simply the first coordinate of the vector $\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}+\cdots+\alpha\_{k} p\_{k}$, ie, the compound lottery. It is analogous for $\operatorname{Prob}\left(u^{2}\right)$ to $\operatorname{Prob}\left(u^{n}\right)$.

From the third to the fourth line: we use again our assumption: $U$ has the expected utility form. Hence we may write $U\left(p\_{1}\right)=u^{1} \cdot p\_{1}^{1}+u^{2} \cdot p\_{1}^{2}+\cdots+u^{n} \cdot p\_{1}^{n}$. It is analogous for $U\left(p\_{2}\right)$ to $U\left(p\_{k}\right)$.

In short:

$$

U\left(\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}+\cdots+\alpha\_{k} p\_{k}\right)=\alpha\_{1} \cdot U\left(p\_{1}\right)+\alpha\_{2} \cdot U\left(p\_{2}\right)+\cdots+\alpha\_{k} \cdot U\left(p\_{k}\right)

$$

That is, $U$ is linear in probabilities, concluding the proof.

QED.

Expected utility form is preserved under positive affine transformations

\section\*{Theorem}

$U, \widetilde{U}$ have expected utility form (and represent the same preferences) $\Leftrightarrow$ there are $\beta>0, \gamma$ such that for all $p, \widetilde{U}(p)=\beta U(p)+\gamma$.

This is MWG proposition 6B2.

Proof

Choose $\bar{p}, \underline{p}$ such that for all lottery $p, \bar{p} \succcurlyeq p \succcurlyeq \underline{p}$.

If $\bar{p} \sim \underline{p}$, then all utility functions are constant, and the result follows immediately. Assume now $\bar{p}>\underline{p}$.

Sufficiency: If $U$ has expected utility form, then $\widetilde{U}(p)=\beta U(p)+\gamma$ also has expected utility form.

Consider a compound lottery $\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}$. That is, we have two lotteries ( $p\_{1}$ and $p\_{2}$ ), and each is chosen with probability $\alpha\_{1}$ and $\alpha\_{2}$, respectively.

Without loss of generality, we consider only two lotteries, but the argument is unchanged for $k$ lotteries.

Compute the utility of this compound lottery under $\widetilde{U}$ :

$$

\begin{gathered}

\widetilde{U}\left(\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}\right)= \\

\beta \cdot U\left(\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}\right)+\gamma= \\

\beta \cdot\left[\alpha\_{1} U\left(p\_{1}\right)+\alpha\_{2} U\left(p\_{2}\right)\right]+\gamma= \\

\alpha\_{1} \beta \cdot U\left(p\_{1}\right)+\alpha\_{2} \beta \cdot U\left(p\_{2}\right)+\underbrace{\left[\alpha\_{1} \gamma+\alpha\_{2} \gamma\right]}\_{=\gamma}= \\

\alpha\_{1} \cdot \underbrace{\left[\beta \cdot U\left(p\_{1}\right)+\gamma\right]}\_{\widetilde{U}\left(p\_{1}\right)}+\alpha\_{2} \cdot \underbrace{\left[\beta \cdot U\left(p\_{2}\right)+\gamma\right]}\_{\widetilde{U}\left(p\_{2}\right)}= \\

\alpha\_{1} \cdot \widetilde{U}\left(p\_{1}\right)+\alpha\_{2} \cdot \widetilde{U}\left(p\_{2}\right)

\end{gathered}

$$

From the first to the second line: we use the definition $\widetilde{U}(p)=\beta U(p)+\gamma$.

From the second to the third line: we use the assumption that $U$ has the expected utility form: hence, $U\left(\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}\right)=\alpha\_{1} U\left(p\_{1}\right)+\alpha\_{2} U\left(p\_{2}\right)$.

From the third to the fourth line: we simply write $\gamma=\alpha\_{1} \gamma+\alpha\_{2} \gamma$, which is true because $\alpha\_{1}+\alpha\_{2}=1$ (it's a probability distribution, so it must sum up to one).

From the fourth to the fifth line: we factor out $\alpha\_{1}$ and $\alpha\_{2}$.

In short:

$$

\widetilde{U}\left(\alpha\_{1} p\_{1}+\alpha\_{2} p\_{2}\right)=\alpha\_{1} \cdot \widetilde{U}\left(p\_{1}\right)+\alpha\_{2} \cdot \widetilde{U}\left(p\_{2}\right)

$$

That is, $\widetilde{U}$ has the expected utility form, as we wanted to show.

Necessity: $U$ and $\widetilde{U}$ have the expected utility form (and represent the same preferences) implies that for some $\beta>0, \gamma$, one has $\widetilde{U}(p)=\beta U(p)+\gamma$

Fix a lottery $p$.

Choose $\lambda\_{p} \in[0,1]$ such that:

$$

U(p)=\lambda\_{p} \cdot U(\bar{p})+\left(1-\lambda\_{p}\right) \cdot U(\underline{p})

$$

This equation has two implications.

First implication: $p \sim \lambda\_{p} \bar{p}+\left(1-\lambda\_{p}\right) \underline{p}$

This holds because we're assuming $U$ has the expected utility form. The previous theorem states that if this is the case, then $U$ is linear in probabilities. Hence the RHS of this last equation may be rewritten as:

$$

\lambda\_{p} \cdot U(\bar{p})+\left(1-\lambda\_{p}\right) \cdot U(\underline{p})=U\left(\lambda\_{p} \cdot \bar{p}+\left(1-\lambda\_{p}\right) \cdot \underline{p}\right)

$$

Hence $U(p)=U\left(\lambda\_{p} \cdot \bar{p}+\left(1-\lambda\_{p}\right) \cdot \underline{p}\right)$. By definition of a utility function, the arguments on each side must be indifferent for the DM.

Second implication:

$$

\lambda\_{p}=\frac{U(p)-U(\underline{p})}{U(\bar{p})-U(\underline{p})}

$$

This is just a rearrangement of the equation above.

We know that $\widetilde{U}$ is linear in probabilities (previous theorems) and represents the same preferences. Hence:

$$

\begin{gathered}

\widetilde{U}(p)=\lambda\_{p} \cdot \widetilde{U}(\bar{p})+\left(1-\lambda\_{p}\right) \cdot \widetilde{U}(\underline{p})= \\

\lambda\_{p} \cdot[\widetilde{U}(\bar{p})-\widetilde{U}(\underline{p})]+\widetilde{U}(\underline{p})= \\

\underbrace{\left.\frac{U(p)-U(\underline{p})}{U(\bar{p})-U(\underline{p})}\right)}\_{\lambda\_{p}} \cdot[\widetilde{U}(\bar{p})-\widetilde{U}(\underline{p})]+\widetilde{U}(\underline{p})

\end{gathered}

$$

In short:

$$

\widetilde{U}(p)=\underbrace{\left(\frac{U(p)-U(\underline{p})}{U(\bar{p})-U(\underline{p})}\right)}\_{\lambda\_{p}} \cdot[\widetilde{U}(\bar{p})-\widetilde{U}(\underline{p})]+\widetilde{U}(\underline{p})

$$

In this last expression, only $U(p)$ depends on $p$. All other terms are parameters. Rearrange this expression to get the following:

$$

\widetilde{U}(p)=\left[\frac{\widetilde{U}(\bar{p})-\widetilde{U}(\underline{p})}{U(\bar{p})-U(\underline{p})}\right] \cdot U(p)+\widetilde{U}(\underline{p})-U(\underline{p}) \cdot\left[\frac{\widetilde{U}(\bar{p})-\widetilde{U}(\underline{p})}{U(\bar{p})-U(\underline{p})}\right]

$$

Again: except $U(p)$, everything in this expression is a parameter, built from functions ( $U$ or $\widetilde{U}$ ) evaluated at specific arguments ( $\bar{p}$ or $\underline{p}$ ). We can label them as we want. Let's choose:

$$

\begin{array}{r}

\beta=\left[\frac{\widetilde{U}(\bar{p})-\widetilde{U}(\underline{p})}{U(\bar{p})-U(\underline{p})}\right] \\

\gamma=\widetilde{U}(\underline{p})-U(\underline{p}) \cdot \underbrace{\left[\frac{\widetilde{U}(\bar{p})-\widetilde{U}(\underline{p})}{U(\bar{p})-U(\underline{p})}\right]}\_{\beta}

\end{array}

$$

Then one has:

$$

\widetilde{U}(p)=\beta U(p)+\gamma

$$

This is what we wanted to show, concluding the proof. QED.

\section\*{Expected utility theorem}

\section\*{Theorem}

(Rational and continuous) Preferences may be represented by an utility function with the expected utility form if and only if it respects the axiom of independence.

Proof of Necessity: if $\succcurlyeq$ respect the axiom of independence, then it may be represented by a utility function with the expected utility form.

Assume there are lotteries $\bar{p}$ and $\underline{p}$ such that for all $p$, one has $\bar{p} \succcurlyeq p \succcurlyeq \underline{p}$.

If $\bar{p} \sim \underline{p}$, the result follows immediately: use a constant utility function.

Assume from now on $\bar{p}>\underline{p}$.

Step 1

Take $\alpha$ and $\beta$ such that $1>\beta>\alpha>0$.

Write:

$$

\begin{gathered}

\bar{p}= \\

\beta \bar{p}+(1-\beta) \bar{p}> \\

\beta \bar{p}+(1-\beta) \underline{p}= \\

(\beta-\alpha) \bar{p}+\alpha \bar{p}+(1-\beta) \underline{p}> \\

(\beta-\alpha) \underline{p}+\alpha \bar{p}+(1-\beta) \underline{p}= \\

\alpha \bar{p}+(1-\alpha) \underline{p}> \\

\alpha \underline{p}+(1-\alpha) \underline{p}= \\

\underline{p}

\end{gathered}

$$

From the first to the second line: $\bar{p}$ is the average of $\bar{p}$ and $\bar{p}$ !

From the second to the third line: we apply the axiom of independence. Observe that we keep $\bar{p}$ in the first term of the sum, but substitute $\bar{p}$ for $\underline{p}$ in the second term. Since $\bar{p}>\underline{p}$, the axiom of independence implies the strict preference.

From the third to the fourth line: add and subtract $\alpha \bar{p}$.

From the fourth to the fifth line: again, we just substitute $\bar{p}$ for $\underline{p}$ in one term of the sum, and leave the rest unchanged. The axiom of independence applies again.

From the fifth to the sixth line: we cancel out $\beta \underline{p}$.

From the sixth to the seventh line: we repeat the argument of the $2^{\text {nd }}$ to $3^{\text {rd }}$ line, in reverse order.

From the seventh to the eighth line: we repeat the argument of the $1^{\text {st }}$ to $2^{\text {nd }}$ line, in reverse order.

Step 2: for all $p$, there is only one $\lambda\_{p}$ such that $\lambda\_{p} \bar{p}+\left(1-\lambda\_{p}\right) \underline{p} \sim p$

Existence follows from continuity. For any lottery $p$, define the sets:

$$

\begin{aligned}

& \{\lambda \in[0,1]: \lambda \bar{p}+(1-\lambda) \underline{p} \succcurlyeq p\} \\

& \{\lambda \in[0,1]: \lambda \bar{p}+(1-\lambda) \underline{p} \preccurlyeq p\}

\end{aligned}

$$

Continuity and completeness of $\succcurlyeq$ imply that both sets are closed (why?). Moreover, any $\lambda$ belongs to at least one of these sets. Since both sets are non-empty and $[0,1]$ is connected, there must by some $\lambda$ belonging to both (again: why?). Define it as $\lambda\_{p}$.

Uniqueness follows from the previous step. If we were to slightly increase the value of $\lambda\_{p}$ (from $\alpha$ to $\beta$ in the notation of the previous step), we would get a new lottery strictly preferred to the DM, breaking indifference.

Step 3: $U(p)=\lambda\_{p}$ is a utility function that represents $\succcurlyeq$

Consider two lotteries $p$ and $q$.

From steps 1 and 2, we may write:

$$

p \succcurlyeq q \Leftrightarrow \lambda\_{\mathrm{p}} \bar{p}+\left(1-\lambda\_{\mathrm{p}}\right) \underline{p} \succcurlyeq \lambda\_{\mathrm{q}} \bar{p}+\left(1-\lambda\_{\mathrm{q}}\right) \underline{p} \Leftrightarrow \lambda\_{\mathrm{p}} \succcurlyeq \lambda\_{\mathrm{q}}

$$

The first $\Leftrightarrow$ comes from step 2: use $p \sim \lambda\_{\mathrm{p}} \bar{p}+\left(1-\lambda\_{\mathrm{p}}\right) \underline{p}$, and analogously $q \sim \lambda\_{\mathrm{q}} \bar{p}+$ $\left(1-\lambda\_{\mathrm{q}}\right) \underline{p}$.

The second $\Leftrightarrow$ comes from step 1 , taking $\lambda\_{\mathrm{p}}=\beta$ and $\lambda\_{\mathrm{q}}=\alpha$.

In short, $p \succcurlyeq q \Leftrightarrow \lambda\_{\mathrm{p}} \succcurlyeq \lambda\_{\mathrm{q}}$. This is the definition of an utility function representing $\succcurlyeq$.

Step 4: $U(p)=\lambda\_{p}$ has the expected utility form.

We have to show that for all $\alpha \in[0,1]$, and for any lotteries $p, p^{\prime}$, one has:

$$

U\left[\alpha p+(1-\alpha) p^{\prime}\right]=\alpha U(p)+(1-\alpha) U\left(p^{\prime}\right)

$$

From step 2, we have:

$$

\begin{gathered}

p \sim \lambda\_{\mathrm{p}} \bar{p}+\left(1-\lambda\_{\mathrm{p}}\right) \underline{p} \\

p^{\prime} \sim \lambda\_{\mathrm{p}^{\prime}} \bar{p}+\left(1-\lambda\_{\mathrm{p}^{\prime}}\right) \underline{p}

\end{gathered}

$$

From step 3, we have : $U(p)=\lambda\_{p}$. These two relations become:

$$

\begin{gathered}

p \sim U(p) \bar{p}+(1-U(p)) \underline{p} \\

p^{\prime} \sim U\left(p^{\prime}\right) \bar{p}+\left(1-U\left(p^{\prime}\right)\right) \underline{p}

\end{gathered}

$$

Take a convex combination $\alpha p+(1-\alpha) p^{\prime}$. Given the two relations above, we have:

$$

\begin{aligned}

& \left.\alpha p+(1-\alpha) p^{\prime} \sim \alpha[U(p) \bar{p}+(1-U(p)) \underline{p}]+(1-\alpha)\left[U\left(p^{\prime}\right) \bar{p}+\left(1-U\left(p^{\prime}\right)\right) \underline{p}\right]\right] \text { Factor out } \bar{p} \text { and } \underline{p} \\

& \sim\left[\alpha U(p)+(1-\alpha) U\left(p^{\prime}\right)\right] \cdot \bar{p}+\left[\alpha(1-U(p))+(1-\alpha)\left(1-U\left(p^{\prime}\right)\right)\right] \cdot \underline{p} \\

& \quad \sim\left[\alpha U(p)+(1-\alpha) U\left(p^{\prime}\right)\right] \cdot \bar{p}+\left[1-\left(\alpha U(p)+(1-\alpha) U\left(p^{\prime}\right)\right)\right] \cdot \underline{p}

\end{aligned}

$$

Notice now that the terms in red are the same. We may denote it by any letter - for example, $\lambda$ (without the subscript to distinguish it from both $\lambda\_{p}$ and $\lambda\_{p^{\prime}}$ ): $\lambda=$ $\alpha U(p)+(1-\alpha) U\left(p^{\prime}\right)$.

The last line becomes:

$$

\lambda \cdot \bar{p}+[1-\lambda] \cdot \underline{p}

$$

In short, and without all the colors:

$$

\alpha p+(1-\alpha) p^{\prime} \sim \lambda \cdot \bar{p}+[1-\lambda] \cdot \underline{p}

$$

But this is the very definition of $\lambda\_{p}$ as defined in step 2 , only applied to $\alpha p+$ $(1-\alpha) p^{\prime}$. (If you want, you may write $\lambda\_{\alpha p+(1-\alpha) p^{\prime}}$ instead of $\lambda$.)

Or, using step 4, we have the more intuitive notation $\lambda=U\left[\alpha p+(1-\alpha) p^{\prime}\right]$.

But we just defined $\lambda=\alpha U(p)+(1-\alpha) U\left(p^{\prime}\right)$.

Since these are the same $\lambda$, one has:

$$

U\left[\alpha p+(1-\alpha) p^{\prime}\right]=\alpha U(p)+(1-\alpha) U\left(p^{\prime}\right)

$$

That is, $U$ has the expected utility property, concluding the proof. QED.

\end{document}

Incerteza e Bens Públicos

\documentclass[10pt]{article}

\usepackage[utf8]{inputenc}

\usepackage[T1]{fontenc}

\usepackage{amsmath}

\usepackage{amsfonts}

\usepackage{amssymb}

\usepackage[version=4]{mhchem}

\usepackage{stmaryrd}

\usepackage{bbold}

\usepackage{graphicx}

\usepackage[export]{adjustbox}

\graphicspath{ {./images/} }

\title{Externalities and Public Goods }

\author{}

\date{}

\begin{document}

\maketitle

\section\*{Simple Bilateral Externality}

Definition: An externality is present whenever the well-being of a consumer or the production possibilities of a firm are directly affected by the actions of another agent in the economy.

Externalities may be positive or negative.

"Directly" exclude any effects that are mediated by prices.

That is, an externality is present if, say, a fishery's productivity is affected by the emissions from a nearby oil refinery, but not simply because the fishery's profitability is affected by the price of oil (which, in turn, is to some degree affected by the oil refinery's output of oil).

The latter type of effect [pecuniary externality] is present in any competitive market but creates no inefficiency.

Indeed, with price-taking behavior, the market is precisely the mechanism that guarantees a Pareto optimal outcome.

This suggests that the presence of an externality is not merely a technological phenomenon but also a function of the set of markets in existence.

Consider partial equilibrium model: no income effect.

Two consumers, $i=1,2$, who constitute a small part of the overall economy.

The actions of these consumers do not affect the prices $p \in \mathbb{R}^{L}$ of the $L$ traded goods in the economy.

At these prices, consumer $i$ 's wealth is $w\_{i}$.

Each consumer has preferences not only over her consumption of the $L$ traded goods $\left(x\_{1 i}, \ldots, x\_{L i}\right)$ but also over some action $\boldsymbol{h} \in \mathbb{R}\_{+}$taken by consumer 1 .

Consumer $i$ 's (differentiable) utility function takes the form $u\_{i}\left(x\_{1 i}, \ldots, x\_{L i}, h\right)$

$$

\text { Assume } \partial u\_{2}\left(x\_{12}, \ldots, x\_{L .2}, h\right) / \partial h \neq 0

$$

Because consumer 1's choice of $h$ affects consumer 2's well-being, it generates an externality.

For example, the two consumers may live next door to each other, and $h$ may be a measure of how loudly consumer 1 plays music.

Or the consumers may live on a river, with consumer 1 further upstream. In this case, $h$ could represent the amount of pollution put into the river by consumer 1 ; more pollution lowers consumer 2's enjoyment of the river.

Define for each consumer $i$ a derived utility function over the level of $h$, assuming optimal commodity purchases by consumer $i$ at prices $p \in \mathbb{R}^{L}$ and wealth $w\_{i}$ :

$$

\begin{array}{ll}

v\_{i}\left(p, w\_{i}, h\right)=\operatorname{Max}\_{x\_{i} \geq 0} & u\_{i}\left(x\_{i}, h\right) \\

& \text { s.t. } p \cdot x\_{i} \leq w\_{i}

\end{array}

$$

Assume that the consumers' utility functions take a quasilinear form with respect to a numeraire commodity.

Derived utility function $v\_{i}(\cdot)$ as $v\_{i}\left(p, w\_{i}, h\right)=\phi\_{i}(p, h)+w\_{i}$

Since prices of the $L$ traded goods are assumed to be unaffected by any of the changes we are considering, we shall suppress the price vector $p$ and simply write $\phi\_{i}(h)$.

We assume that $\phi\_{i}(\cdot)$ is twice differentiable with $\phi\_{i}^{\prime \prime}(\cdot)<0$.

Everything we do here applies if agents are firms (or one firm and one consumer).

\section\*{Nonoptimality of the Competitive Outcome}

Consider a competitive equilibrium in which commodity prices are $p$.

That is, in equilibrium, each of the two consumers maximizes her utility limited only by her wealth and the prices $p$ of the traded goods.

Consumer 1 chooses her level of $h \geq 0$ to maximize $\phi\_{1}(h)$.

Assume throughout interior solutions: now, $h^{\*}>0$.

The equilibrium level of $h, h^{\*}$, satisfies the necessary and sufficient first-order condition

$$

\phi\_{1}^{\prime}\left(h^{\*}\right)=0

$$

Pareto optimal allocation: maximize the joint surplus of the two consumers:

$$

\operatorname{Max}\_{h \geq 0} \phi\_{1}(h)+\phi\_{2}(h)

$$

Necessary and sufficient first-order condition for $h^{\circ}$ (assume strictly positive):

$$

\phi\_{1}^{\prime}\left(h^{\circ}\right)=-\phi\_{2}^{\prime}\left(h^{\circ}\right)

$$

When external effects are present, so that $\phi\_{2}^{\prime}(h) \neq 0$ at all $h$, the equilibrium level of $h$ is not optimal (unless $h^{\circ}=h^{\*}=0$ ).

If $\phi\_{2}^{\prime}(\cdot)<0$ (negative externality), then $\phi\_{1}^{\prime}\left(h^{\circ}\right)=-\phi\_{2}^{\prime}\left(h^{\circ}\right)>0=\phi\_{1}^{\prime}\left(h^{\*}\right)$

That is, $\phi\_{1}^{\prime}\left(h^{\circ}\right)>\phi\_{1}^{\prime}\left(h^{\*}\right)$

$\phi\_{1}^{\prime}$ decreasing implies $h^{\*}>h^{\circ}$.

Analogously, $\phi\_{2}^{\prime}(\cdot)>0$ (positive externality) implies $h^{\*}<h^{\circ}$.

\begin{center}

\includegraphics[max width=\textwidth]{2024\_07\_12\_a816caed4581474ed3fag-04}

\end{center}

Figure 11.B.1: The equilibrium ( $h^{\*}$ ) and optimal ( $h^{\circ}$ ) levels of a negative externality.

Figure 11.B.1 depicts the solution for a case in which $h$ constitutes a negative external effect, so that $\phi\_{2}^{\prime}(h)<0$ at all $h$.

In the figure, we graph $\phi\_{1}^{\prime}(\cdot)$ and $-\phi\_{2}^{\prime}(\cdot)$.

The competitive equilibrium level of the externality $h^{\*}$ occurs at the point where the graph of $\phi\_{1}^{\prime}(\cdot)$ crosses the horizontal axis.

In contrast, the optimal externality level $h^{\circ}$ corresponds to the point of intersection between the graphs of the two functions.

Optimality does not usually entail the complete elimination of a negative externality.

Externality's level is adjusted to the point where the marginal benefit to consumer 1 of an additional unit of the externality-generating activity, $\phi\_{1}^{\prime}\left(h^{\circ}\right)$, equals its marginal cost to consumer $2,-\phi\_{2}^{\prime}\left(h^{\circ}\right)$.

In the current example, quasilinear utilities lead the optimal level of the externality to be independent of the consumers' wealth levels.

In the absence of quasi-linearity, wealth effects for the consumption of the externality make its optimal level depend on the consumers' wealth levels.

When the agents under consideration are firms, wealth effects are always absent.

\section\*{Traditional Solutions to the Externality Problem}

Quotas and taxes

Suppose that $h$ generates a negative external effect, so that $h^{\circ}<h^{\*}$.

Government can mandate that $h$ be no larger than $h^{\circ}$, its optimal level.

With this constraint, consumer 1 will indeed fix the level of the externality at $h^{\circ}$.

Second option: tax on the externality-generating activity.

Pigouvian taxation.

\begin{center}

\includegraphics[max width=\textwidth]{2024\_07\_12\_a816caed4581474ed3fag-06}

\end{center}

Figure 11.B.2: The optimality restoring Pigouvian tax.

Suppose that consumer 1 is made to pay a tax of $t\_{h}$ per unit of $h$.

Define $t\_{h}=-\phi\_{2}^{\prime}\left(h^{\circ}\right)>0$

Consumer 1 will then choose the level of $h$ that solves

$$

\operatorname{Max}\_{h \geq 0} \phi\_{1}(h)-t\_{h} h

$$

Necessary and sufficient first-order condition:

$$

\phi\_{1}^{\prime}(h)=t\_{h}

$$

But $t\_{h}=-\phi\_{2}^{\prime}\left(h^{\circ}\right)$

Hence $\phi\_{1}^{\prime}(h)=t\_{h}=-\phi\_{2}^{\prime}(h)$

This is the condition for optimality.

Moreover, given $\phi\_{1}^{\prime \prime}(\cdot)<0, h^{\circ}$ must be the unique solution to problem.

Figure 11.B.2 illustrates this solution for a case in which $h^{\circ}>0$.

The optimality-restoring tax is exactly equal to the marginal externality at the optimal solution.

That is, it is exactly equal to the amount that consumer 2 would be willing to pay to reduce $h$ slightly from its optimal level $h^{\prime}$.

When faced with this tax, consumer 1 is effectively led to carry out an individual cost benefit computation that internalizes the externality that she imposes on consumer 2 .

Analogous for positive externality with $t\_{h}=-\phi\_{2}^{\prime}\left(h^{\prime}\right)<0$

$t\_{h}$ tis a per-unit subsidy.

Comments about Pigouvian solution:

First, we can actually achieve optimality either by taxing the externality or by subsidizing its reduction.

Consider, for example, the case of a negative externality.

Suppose the government pays a subsidy of $s\_{h}=-\phi\_{2}^{\prime}\left(h^{\circ}\right)>0$ for every unit that consumer 1's choice of $h$ is below $h^{\*}$, its level in the competitive equilibrium.

If so, then consumer 1 will maximize $\phi\_{1}(h)+s\_{h}\left(h^{\*}-h\right)=\phi\_{1}(h)-t\_{h} h+t\_{h} h^{\*}$.

But this is equivalent to a tax of $t\_{h}$ per unit on $h$ combined with a lump-sum payment of $t\_{h} h^{\*}$.

Hence, a subsidy for the reduction of the externality can replicate the outcome of the tax.

Distribution is different, but in principle may be combined with a lumpsum transfer

Second, in general, it is essential to tax the externality-producing activity directly.

Common example of this sort arises when a firm pollutes in the process of producing output.

A tax on its output leads the firm to reduce its output level but may not have any effect (or, more generally, may have too little effect) on its pollution emissions.

Taxing output achieves optimality only in the special case in which emissions bear a fixed monotonic relationship to the level of output.

In this special case, emissions can be measured by the level of output, and a tax on output is essentially equivalent to a tax on emissions.

Third, the tax/subsidy and the quota approaches are equally effective in achieving an optimal outcome.

However, the government must have a great deal of information about the benefits and costs of the externality to set the optimal levels of either the quota or the tax.

When the government does not possess this information the two approaches typically are not equivalent.

Fostering bargaining over externalities: enforceable property rights

Another approach to the externality problem aims at a less intrusive form of intervention:

Ensure that conditions are met for the parties to reach an optimal agreement on the level of the externality.

Suppose that we establish enforceable property rights with regard to the externalitygenerating activity.

For example, that we assign the right to an "externality-free" environment to consumer 2.

Then consumer 1 is unable to engage in the externality-producing activity without consumer 2's permission.

For simplicity, imagine that the bargaining between the parties takes a form in which consumer 2 makes consumer 1 a take-it-or-leave-it offer, demanding a payment of $T$ in return for permission to generate externality level $h$.

Consumer 1 will agree to this demand if and only if she will be at least as well off as she would be by rejecting it.

That is, if and only if $\phi\_{1}(h)-T \geq \phi\_{1}(0)$.

Hence, consumer 2 will choose her offer $(h, T)$ to solve

$$

\begin{array}{ll}

\operatorname{Max}\_{h \geq 0, T} & \phi\_{2}(h)+T \\

& \text { s.t. } \phi\_{1}(h)-T \geq \phi\_{1}(0)

\end{array}

$$

The constraint is binding in any solution to this problem. In particular:

$$

T=\phi\_{1}(h)-\phi\_{1}(0)

$$

Therefore, consumer 2's optimal offer involves the level of $h$ that solves

$$

\operatorname{Max}\_{h \geq 0} \phi\_{2}(h)+\phi\_{1}(h)-\phi\_{1}(0)

$$

The solution is precisely $h^{\circ}$, the socially optimal level.

The precise allocation of these rights between the two consumers is inessential to the achievement of optimality.

Consumer 1 may have the right to generate as much of the externality as she wants.

In the absence of any agreement, consumer 1 will generate externality level $h^{\*}$.

Now consumer 2 will need to offer a $T<0$ (i.e., to pay consumer 1) to have $h<h^{\*}$.

In particular, consumer 1 will agree to externality level $h$ if and only if:

$$

\phi\_{1}(h)-T \geq \phi\_{1}\left(h^{\*}\right)

$$

As a consequence, consumer 2 will offer to set $h$ at the level that solves:

$$

\operatorname{Max}\_{h}\left(\phi\_{2}(h)+\phi\_{1}(h)-\phi\_{1}\left(h^{\*}\right)\right)

$$

Once again, the optimal externality level $h^{\circ}$ results.

The allocation of rights affects only the final wealth of the two consumers by altering the payment made by consumer 1 to consumer 2.

In the first case, consumer 1 pays $\phi\_{1}\left(h^{\prime \prime}\right)-\phi\_{1}(0)>0$ to be allowed to set $h^{\circ}>0$

In the second, she "pays" $\phi\_{1}\left(h^{\prime \prime}\right)-\phi\_{1}\left(h^{\*}\right)<0$ in return for setting $h^{\circ}<h^{\*}$.

\begin{center}

\includegraphics[max width=\textwidth]{2024\_07\_12\_a816caed4581474ed3fag-10}

\end{center}

Figure 11.B.3: The final distribution of utilities under different property rights institutions and different bargaining procedures.

This is an instance of what is known as the Coase theorem [for Coase (1960)]:

\section\*{If trade of the externality can occur, then bargaining will lead to an efficient outcome no matter how property rights are allocated.}

Moreover, no income effect => efficiency and distribution problems can be separated: efficient level of externality does not depend on allocation of property rights.

The existence of both well-defined and enforceable property rights is essential for this type of bargaining to occur.

If property rights are not well defined, it will be unclear whether consumer 1 must gain consumer 2's permission to generate the externality.

If property rights cannot be enforced (perhaps the level of $h$ is not easily measured), then consumer 1 has no need to purchase the right to engage in the externality-generating activity from consumer 2 .

For this reason, proponents of this type of approach focus on the absence of these legal institutions as a central impediment to optimality.

This solution to the externality problem has a significant advantage over the tax and quota schemes in terms of the level of knowledge required of the government.

The consumers must know each other's preferences, but the government need not.

For bargaining over the externality to lead to efficiency, it is important that the consumers know this information.

When the agents are to some extent ignorant of each others' preferences, bargaining need not lead to an efficient outcome.

Two further points:

First, in the case in which the two agents are firms, one form that an efficient bargain might take is the sale of one of the firms to the other.

The resulting merged firm would then fully internalize the externality in the process of maximizing its profits.

This conclusion presumes that the owner of a firm has full control over all its functions. In more complicated (but realistic) settings in which this is not true, say because owners must hire managers whose actions cannot be perfectly controlled, the results of a merger and of an agreement over the level of the externality need not be the same.

See Holmstrom and Tirole (1989) for a discussion of these issues in the theory of the firm.

Second, note that all three approaches require that the externality-generating activity be measureable.

This is not a trivial requirement; in many cases, such measurement may be either technologically infeasible or very costly (consider the cost of measuring air pollution or noise).

A proper computation of costs and benefits should take these costs into account. If measurement is very costly, then it may be optimal to simply allow the externality to persist.

\section\*{Externalities and Missing Markets}

\section\*{There is a connection between externalities and missing markets.}

A market system can be viewed as a particular type of trading procedure (which is just a form of social interaction).

Suppose that property rights are well defined and enforceable and that a competitive market for the right to engage in the externality-generating activity exists.

For simplicity, assume that consumer 2 has the right to an externality-free environment.

Let $p\_{h}$ denote the price of the right to engage in one unit of the activity.

In choosing how many of these rights to purchase, say $h\_{1}$, consumer 1 will solve

$$

\operatorname{Max}\_{h\_{1} \geq 0} \phi\_{1}\left(h\_{1}\right)-p\_{h} h\_{1}

$$

which has the first-order condition

$$

\phi\_{1}^{\prime}\left(h\_{1}\right)=p\_{h}

$$

In deciding how many rights to sell, $h\_{2}$, consumer 2 will solve

$$

\operatorname{Max}\_{h\_{2} \geq 0} \phi\_{2}\left(h\_{2}\right)+p\_{h} h\_{2}

$$

which has the first-order condition

$$

\phi\_{2}^{\prime}\left(h\_{2}\right)=-p\_{h}

$$

In a competitive equilibrium, the market for these rights must clear: $h\_{1}=h\_{2}$.

Hence, the level of rights traded in this competitive rights market satisfies

$$

\phi\_{1}^{\prime}(h)=-\phi\_{2}^{\prime}(h)

$$

This is the optimal level $h=h^{\circ}$.

The equilibrium price of the externality is $p\_{h}^{\*}=\phi\_{1}^{\prime}\left(h^{\circ}\right)=-\phi\_{2}^{\prime}\left(h^{\circ}\right)$.

Consumer 1 and 2's equilibrium utilities are then $\phi\_{1}\left(h^{\circ}\right)-p\_{h}^{\*} h^{\circ}$ and $\phi\_{2}\left(h^{\circ}\right)+p\_{h}^{\*} h^{\circ}$, respectively.

\section\*{The market therefore works as a particular bargaining procedure for splitting}

the gains from trade.If a competitive market exists for the externality, then optimality results.

\section\*{Externalities can be seen as being inherently tied to the absence of certain competitive markets.}

Indeed, our definition of an externality explicitly required that an action chosen by one agent must directly affect the well-being or production capabilities of another.

Once a market exists for an externality, however, each consumer decides for herself how much of the externality to consume at the going prices.

The idea of a competitive market for the externality in the present example is rather unrealistic.

In a market with only one seller and one buyer, price taking would be unlikely.

However, most important externalities are produced and felt by many agents.

Thus, we might hope that in these multilateral settings, price taking would be a more reasonable assumption and, as a result, that a competitive market for the externality would lead to an efficient outcome.

\section\*{Public Goods}

Definition: A public good is a commodity for which use of a unit of the good by one agent does not preclude its use by other agents.

Public goods are nondepletable:

Consumption by one individual does not affect the supply available for other individuals.

Knowledge provides a good illustration.

The use of a piece of knowledge for one purpose does not preclude its use for others.

Commodities studied up to this point have been assumed to be of a private, or depletable, nature;

A distinction can also be made according to whether exclusion of an individual from the benefits of a public good is possible.

\section\*{Every private good is automatically excludable, but public goods may or may not be.}

The patent system, for example, is a mechanism for excluding individuals (although imperfectly) from the use of knowledge developed by others.

On the other hand, it might be technologically impossible, or at the least very costly, to exclude some consumers from the benefits of national defense or of a project to improve air quality.

Focus here on the case in which exclusion is not possible.

A public "good" need not necessarily be desirable; that is, we may have public bads (e.g., foul air).

In this case, we should read the phrase "does not preclude" to mean "does not decrease."

\section\*{Conditions for Pareto Optimality}

Consider a setting with $I$ consumers and one public good, in addition to $L$ traded goods of the usual, private, kind.

Partial equilibrium perspective: the quantity of the public good has no effect on the prices of the $L$ traded goods and that each consumer's utility function is quasilinear with respect to the same numeraire, traded commodity.

We can therefore define, for each consumer $i$, a derived utility function over the level of the public good.

Let $x$ denote the quantity of the public good.

Denote consumer $i$ 's utility from the public good by $\phi\_{i}(x)$.

Assume that this function is twice differentiable, with $\phi\_{i}^{\prime \prime}(x)<0$ at all $x \geq 0$.

Precisely because we are dealing with a public good, the argument $x$ does not have an $i$ subscript.

The cost of supplying $q$ units of the public good is $c(q)$.

Assume that $c(\cdot)$ is twice differentiable, with $c^{\prime \prime}(q)>0$ at all $q \geq 0$.

Take $\phi\_{i}^{\prime}(\cdot)>0$ for all $i$ and $c^{\prime}(\cdot)>0$ : wlog, public good is desirable and costly.

In this quasilinear model, any Pareto optimal allocation must maximize aggregate surplus.

Therefore must involve a level of the public good that solves

$$

\operatorname{Max}\_{q \geq 0} \sum\_{i=1}^{I} \phi\_{i}(q)-c(q)

$$

The necessary and sufficient first-order condition for the optimal quantity $q^{\circ}$ is then

$$

\sum\_{i=1}^{I} \phi\_{i}^{\prime}\left(q^{\circ}\right)=c^{\prime}\left(q^{\circ}\right)

$$

This condition is the classic optimality condition for a public good first derived by Samuelson $(1954 ; 1955)$.

\section\*{At the optimal level of the public good the sum of consumers' marginal benefits from the public good is set equal to its marginal cost.}

For a private good, where each consumer's marginal benefit from the good is equated to its marginal cost.

\section\*{Inefficiency of Private Provision of Public Goods}

Consider a public good provided by means of private purchases by consumers.

We imagine that a market exists for the public good and that each consumer $i$ chooses how much of the public good to buy, denoted by $x\_{i} \geq 0$, taking as given its market price $p$.

The total amount of the public good purchased by consumers is then $x=\sum\_{i} x\_{i}$.

Formally, we treat the supply side as consisting of a single profit-maximizing firm with cost function $c(\cdot)$ that chooses its production level taking the market price as given.

(We can also think of the supply behavior of this firm as representing the industry supply of $J$ price-taking firms whose aggregate cost function is $c(\cdot)$.

At a competitive equilibrium involving price $p^{\*}$, each consumer $i$ 's purchase of the public good $x\_{i}^{\*}$ must maximize her utility:

$$

\operatorname{Max}\_{x\_{i} \geq 0} \phi\_{i}\left(x\_{i}+\sum\_{k \neq i} x\_{k}^{\*}\right)-p^{\*} x\_{i}

$$

In determining her optimal purchases, consumer $i$ takes as given the amount of the private good being purchased by each other consumer.

There is a bit of game theory here: this is how we find a Nash equilibrium.

Consumer $i$ 's purchases $x\_{i}^{\*}$ must therefore satisfy the necessary and sufficient first-order condition

$$

\phi\_{i}^{\prime}\left(x\_{i}^{\*}+\sum\_{k \neq i} x\_{k}^{\*}\right)=p^{\*}

$$

Letting $x^{\*}=\sum\_{i} x\_{i}^{\*}$ denote the equilibrium level of the public good, for each consumer $i$ we must therefore have (for $x\_{i}^{\*}>0$ )

$$

\phi\_{i}^{\prime}\left(x^{\*}\right)=p^{\*}

$$

The firm's supply $q^{\*}$, on the other hand, must solve:

$$

\operatorname{Max}\_{q \geq 0}\left(p^{\*} q-c(q)\right)

$$

and therefore must satisfy the standard necessary and sufficient first-order condition

$$

p^{\*}=c^{\prime}\left(q^{\*}\right)

$$

At a competitive equilibrium, $q^{\*}=x^{\*}$

Hence:

$$

\phi\_{i}^{\prime}\left(q^{\*}\right)=p^{\*}=c^{\prime}\left(q^{\*}\right)

$$

Or simply:

$$

\phi\_{i}^{\prime}\left(q^{\*}\right)=c^{\prime}\left(q^{\*}\right)

$$

Hence:

$$

\sum\_{i}\left[\phi\_{i}^{\prime}\left(q^{\*}\right)-c^{\prime}\left(q^{\*}\right)\right]=0

$$

Recalling that $\phi\_{i}^{\prime}(\cdot)>0$ and $c^{\prime}(\cdot)>0$, this implies that whenever $I>1$ and $q^{\*}>0$ we have

$$

\sum\_{i=1}^{I} \phi\_{i}^{\prime}\left(q^{\*}\right)>c^{\prime}\left(q^{\*}\right)

$$

Whenever $q^{\circ}>0$ and $I>1$, the level of the public good provided is too low; that is, $q^{\*}<$ $q^{0}$.

\begin{center}

\includegraphics[max width=\textwidth]{2024\_07\_12\_a816caed4581474ed3fag-18}

\end{center}

Figure 11.C.1: Private provision leads to an insufficient level of a desirable public good.

The cause of this inefficiency can be understood in terms of our discussion of externalities.

Each consumer's purchase of the public good provides a direct benefit not only to the consumer herself but also to every other consumer.

Hence, private provision creates a situation in which externalities are present.

The failure of each consumer to consider the benefits for others of her public good provision is often referred to as the free-rider problem:

Each consumer has an incentive to enjoy the benefits of the public good provided by others while providing it insufficiently herself.

In the present model, the free-rider problem takes a very stark form.

To see this most simply, suppose that we can order the consumers according to their marginal benefits, in the sense that $\phi\_{1}^{\prime}(x)<\cdots<\phi\_{I}^{\prime}(x)$ at all $x \geq 0$.

Then optimality can hold with equality only for a single consumer and, moreover, this must be the consumer labeled $I$.

\section\*{Therefore, only the consumer who derives the largest (marginal) benefit from the public good will provide it; all others will set their purchases equal to zero in the equilibrium.}

The equilibrium level of the public good is then the level $q^{\*}$ that satisfies $\phi\_{I}^{\prime}\left(q^{\*}\right)=c^{\prime}\left(q^{\*}\right)$.

Figure 11.C. 1 depicts both this equilibrium and the Pareto optimal level. Note that the curve representing $\sum\_{i} \phi\_{i}^{\prime}(q)$ geometrically corresponds to a vertical summation of the individual curves representing $\phi\_{i}(q)$ for $i=1, \ldots, I$.

(Whereas in the case of a private good, the market demand curve is identified by adding the individual demand curves horizontally).

The inefficiency of private provision is often remedied by governmental intervention in the provision of public goods.

Just as with externalities, this can happen not only through quantity-based intervention (such as direct governmental provision) but also through "price-based" intervention in the form of taxes or subsidies.

For example, suppose that there are two consumers with benefit functions $\phi\_{1}\left(x\_{1}+x\_{2}\right)$ and $\phi\_{2}\left(x\_{1}+x\_{2}\right)$, where $x\_{i}$ is the amount of the public good purchased by consumer $i$, and that $q^{\circ}>0$.

A subsidy to each consumer $i$ per unit purchased of $s\_{i}=\phi\_{-i}^{\prime}\left(q^{\circ}\right)$ faces each consumer with the marginal external effect of her actions and so generates an optimal level of public good provision by consumer $i$.

Formally, if ( $\left.\tilde{x}\_{1}, \tilde{x}\_{2}\right)$ are the competitive equilibrium levels of the public good purchased by the two consumers given these subsidies, and if $\tilde{p}$ is the equilbrium price, then consumer $i$ 's purchases of the public good, $\tilde{x}\_{i}$, must solve:

$$

\operatorname{Max}\_{x\_{i} \geq 0} \phi\_{i}\left(x\_{i}+\tilde{x}\_{j}\right)+s\_{i} x\_{i}-\tilde{p} x\_{i}

$$

and so $\tilde{x}\_{i}$ must satisfy the necessary and sufficient first-order condition

$$

\phi\_{i}^{\prime}\left(\tilde{x}\_{1}+\tilde{x}\_{2}\right)+s\_{i} \leq \tilde{p} \text {, with equality of } \tilde{x}\_{i}>0

$$

Substituting for $s\_{i}$, and using the fact that price equals marginal cost and the marketclearing condition that $\tilde{x}\_{1}+\tilde{x}\_{2}=\tilde{q}$, we conclude that $\tilde{q}$ is the total amount of the public good in the competitive equilibrium given these subsidies if and only if

$$

\phi\_{i}^{\prime}(\tilde{q})+\phi\_{-i}^{\prime}\left(q^{\circ}\right) \leq c^{\prime}(\tilde{q})

$$

with equality for some $i$ if $\tilde{q}>0$.

Use $\sum\_{i=1}^{l} \phi\_{i}^{\prime}\left(q^{\circ}\right) \leq c^{\prime}\left(q^{\circ}\right)$ to see that $\tilde{q}=q^{\circ}$.

Note that both optimal direct public provision and this subsidy scheme require that the government know the benefits derived by consumers from the public good.

I.e., their willingness to pay in terms of private goods.

\section\*{Lindahl Equilibria}

Although private provision of the sort studied above results in an inefficient level of the public good, there is in principle a market institution that can achieve optimality.

Suppose that, for each consumer $i$, we have a market for the public good "as experienced by consumer $i . "$

That is, we think of each consumer's consumption of the public good as a distinct commodity with its own market.

We denote the price of this personalized good by $p\_{i}$.

Note that $p\_{i}$ may differ across consumers.

Suppose also that, given the equilibrium price $p\_{i}^{\* \*}$, each consumer $i$ sees herself as deciding on the total amount of the public good she will consume, $x\_{i}$, so as to solve

$$

\operatorname{Max}\_{x\_{i}>0} \phi\_{i}\left(x\_{i}\right)-p\_{i}^{\* \*} x\_{i}

$$

Her equilibrium consumption level $x\_{i}^{\* \*}$ must therefore satisfy the necessary and sufficient first-order condition

$$

\phi\_{i}^{\prime}\left(x\_{i}^{\* \*}\right) \leq p\_{i}^{\* \*}, \text { with equality if } x\_{i}^{\* \*}>0

$$

The firm is now viewed as producing a bundle of $I$ goods with a fixed-proportions technology (i.e., the level of production of each personalized good is necessarily the same).

Thus, the firm solves

$$

\operatorname{Max}\_{q \geq 0}\left(\sum\_{i=1}^{1} p\_{i}^{\* \*} q\right)-c(q)

$$

The firm's equilibrium level of output $q^{\* \*}$ therefore satisfies the necessary and sufficient first-order condition

$$

\sum\_{i=1}^{I} p\_{i}^{\* \*} \leq c^{\prime}\left(q^{\* \*}\right), \text { with equality if } q^{\* \*}>0

$$

We have then:

$$

\sum\_{i=1}^{I} \phi\_{i}^{\prime}\left(q^{\* \*}\right) \leq c^{\prime}\left(q^{\* \*}\right), \text { with equality if } q^{\* \*}>0

$$

The equilibrium level of the public good consumed by each consumer is exactly the efficient level: $q^{\* \*}=q^{\circ}$.

This type of equilibrium in personalized markets for the public good is known as a Lindahl equilibrium, after Lindahl (1919).

To understand why we obtain efficiency, note that once we have defined personalized markets for the public good, each consumer, taking the price in her personalized market as given, fully determines her own level of consumption of the public good.

\section\*{Externalities are eliminated.}

Yet, despite the attractive properties of Lindahl equilibria, their realism is questionable.

Note, first, that the ability to exclude a consumer from use of the public good is essential if this equilibrium concept is to make sense.

Otherwise a consumer would have no reason to believe that in the absence of making any purchases of the public good she would get to consume none of it.

Moreover, even if exclusion is possible, these are markets with only a single agent on the demand side.

As a result, price-taking behavior of the sort presumed is unlikely to occur.

The idea that inefficiencies can in principle be corrected by introducing the right kind of markets is a very general one.

In particular cases, however, this "solution" may or may not be a realistic possibility.

\end{document}